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QUALITATIVE INVESTIGATION OF A DYNAMIC SYSTEM

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We make a qualitative investigation of a dynamic system by bifurcation-theoretic methods [1], using the property of the monotonic rotation of the direction field. We trace the possible bifurcations and the behavior of the bifurcation curves in various sections of the parameter space. The system has been examined before [2, 3], however, a complete qualitative investigation has not been made.

1. Rotation of the field. We examine the system

$$\frac{d\varphi}{dt} = y = P, \qquad \frac{dy}{dt} = \beta - \sin\varphi - 2\alpha s \frac{y}{s^2 + y^2} = Q \tag{1.1}$$

for positive α, β and s. The difference between the direction fields of system (1.1) with parameters β, α_0, s_0 and of an altered system with parameters β, α_1, s_1 for $y \neq 0$ is

$$2\,[s_0 s_1(\alpha_1 s_0 - \alpha_0 s_1) + (\alpha_1 s_1 - \alpha_0 s_0)\,y^2]\,[(s_0^2 + y^2)(s_1^2 + y^2)]^{-1} \tag{1.2}$$

For a fixed β a monotonic rotation is realized if the altered parameter values α_1 and s_1 are chosen so as to fulfill the condition

$$(\alpha_1 s_0 - \alpha_0 s_1)(\alpha_1 s_1 - \alpha_0 s_0) \geqslant 0$$

In particular, a monotonic rotation is realized when α and s are varied along the k-curves ($\alpha s = k$, $0 < k < \infty$) or the $\varkappa$-curves ($\alpha / s = \varkappa$, $0 < \varkappa < \infty$). The families of k- and $\varkappa$-curves cover, each separately, the whole part being examined of the αs-plane. The curves of the altered and original systems intersect on the straight line $y = 0$ with tangency on the φ-axis. As β varies the field of directions on the lower and upper half-cylinders rotate in opposite directions. In this case the straight line $y = 0$ is the contact curve.

2. Qualitative structures at the end-points of the k-curves. In order to observe the change in the qualitative structure of the phase space under a monotonic rotation of the field directions with the parameters varying along the k-curves, we need to know the structures of the partitioning of the phase space by the end-points of the k-curves for small and for large s (and, respectively, for large and small α). In a cylindrical phase space (on the strip $-\pi \leqslant \varphi \leqslant \pi$ with the edges identified) the equilibrium states are O_1 $(\arcsin \beta, 0)$, a stable focus or node, and O_2 $(\pi - \arcsin \beta, 0)$, a saddle. The directions by which the trajectories of system (1.1) enter into the saddle are determined by the equation

$$\zeta^2 + \frac{2\alpha}{s}\zeta - \sqrt{1 - \beta^2} = 0$$

For $0 \leqslant \beta \leqslant 1$ one of the roots is always negative and corresponds to the direction by which an ω-separatrix enters into the saddle. Suppose that on some straight line $\varphi = \varphi_0$ we mark, in the interval $(\arcsin \beta, \pi - \arcsin \beta)$ between the singular points, the coordinate η_0 of the point of intersection of the straight line with the ω-separatrix of the saddle. If as s decreases we move in the parameter space along the k-curves, then the vector field will rotate monotonically clockwise and η_0 will grow. At the same time, on the lower branch (having positive ordinate values only outside the interval $(\arcsin \beta, \pi - \arcsin \beta)$) of the isocline of horizontal slopes on the upper half-cylinder, the maximum, equal to

$$y_1 = [k - \sqrt{k^2 - s^2(\beta + 1)^2}]\,(\beta + 1)^{-1}$$

for $\varphi = -\pi/2$, will decrease unboundedly. Therefore, for any k we can select s such that the inequality $y_1 < \eta_0$ is fulfilled, and then the ω-separatrix enters the saddle, twisting away from the upper half-cylinder. For $\beta > 0$ the point at infinity is stable on the upper half-cylinder. Indeed, if for large $y > 0$ we set $y = 1 / \rho$ and we construct in the usual way the successor function in the neighborhood of a small $\rho = \rho_0$ [4], we obtain

$$\rho_1(2\pi) - \rho_0(0) = -2\pi\beta\rho_0^3 + 4\pi\alpha s\rho_0^4 + \ldots$$

Hence follows the existence of at least one unstable limit cycle situated above the minimum of the upper branch of the isocline positive slopes, i.e., for

$$y > y_2 = [k + \sqrt{k^2 - s^2(\beta + 1)^2}]\,(\beta + 1)^{-1}$$

Since for small s the curve $P_\varphi' + Q_y' \equiv 2k\,(y^2 - s^2)(y^2 + s^2)^{-2} = 0$ does not intersect the upper branch of the isocline of horizontal slopes ($y_2 > s$ always for small s), this cycle is unique. For small s the curve $P_\varphi' + Q_y' = 0$ also does not intersect the lower branch of the isocline of horizontal slopes, therefore, limit cycles cannot exist around the point O_1.

Limit cycles cannot exist also on the lower half-cylinder. System (1.1) is equivalent to the equation

$$y\,dy + \sin\varphi\,d\varphi = \left(\beta - 2\alpha s\frac{y}{s^2+y^2}\right)d\varphi$$

Therefore, for a closed contour girding the cylinder and made up from the trajectories of system (1.1), we have

$$\int_{-\pi}^{\pi}\left(\beta - 2\alpha s\frac{y}{s^2+y^2}\right)d\varphi = 0$$

but this is impossible for $y < 0$ and for positive β, α and s.

The qualitative pattern of the phase space for sufficiently small s on any curve $\alpha s = k$ is shown in Fig. 1 (*1*).

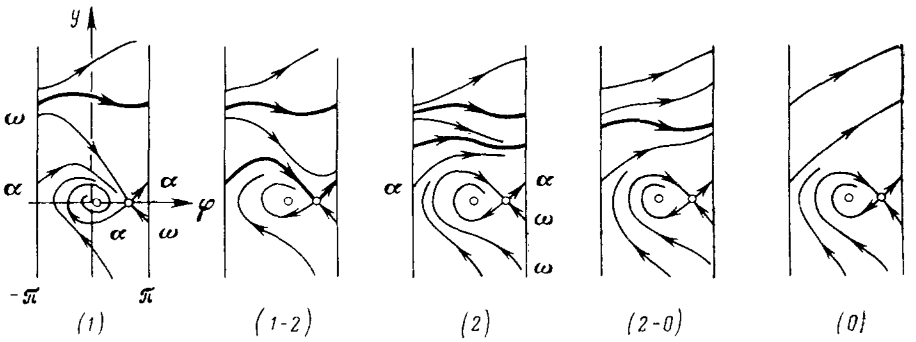

Fig. 1.

Let us observe the behavior of the saddle's α-separatrices for large s. We consider two conservative comparison systems

$$d\varphi\,/\,dt = y, \qquad dy\,/\,dt = \beta - \sin\varphi \tag{2.1}$$

$$d\varphi\,/\,dt = y, \qquad dy\,/\,dt = \beta - \sin\varphi - k\,/\,s \tag{2.2}$$

(the condition $0 < \beta - k\,/\,s < 1$ is fulfilled for any k at large s). For system (2.1) the point O_1 ($\arcsin\beta$, 0) is an equilibrium state of the center type, and the separatrices of the saddle O_2 ($\pi - \arcsin\beta$, 0) form a loop around O_1. The field of directions of system (1.1) is turned clockwise relative to system (2.1). Therefore, the α-separatrix of the saddle of system (1.1), going onto the upper half-cylinder, enters the point O_1.

On the upper half-cylinder the trajectories of system (2.2) are spirals winding around the cylinder and going off to infinity. On the upper half-cylinder the field directions of system (1.1) is turned counterclockwise relative to system (2.2) everywhere excepting the straight line $y = s$ (there is tangency with the intersection on the straight line $y = s$). Therefore, the α-separatrix of the saddle of system (1.1), going onto the upper half-cylinder, cannot intersect the α-separatrix of the saddle of system (2.2), issuing from the saddle O ($\pi - \arcsin(\beta - k\,/\,s)$, 0) located to the right of the saddle O_2 ($\pi - \arcsin\beta$, 0), and must go off to infinity. There are no limit cycles. The behavior of the α-separatrices completely determines the qualitative pattern of the phase space partitioning. The qualitative pattern on any curve $\alpha s = k$ for sufficiently large s is shown in Fig. 1 (*0*).

3. Qualitative patterns of the phase space and possible bifurcations for $0 < \beta < 1$. The k-curves join the parameter space regions corresponding to the structures shown on Figs. 1 (*1*) and 1(*0*). As s increases along the k-curves the points P_1 and P_2 of intersection of the straight line $\varphi = \arcsin \beta$ with the α- and ω-separatrices of the saddle on the upper half-cylinder come together, coincide for some value $s = s_0(k)$ (respectively, $\alpha = \alpha_0(k)$) and then diverge monotonically. The sets of points $s_0(k)$, $\alpha_0(k)$, corresponding to a structurally unstable bifurcation structure, for which the α- and ω-separatrices of the saddle form a loop on the upper half-cylinder (P_1 and P_2 coincide) form a continuous curve L. Every k-curve intersects curve L at one point. As s passes through the value corresponding to an intersection of a k-curve and curve L there arises and then collapses a separatrix loop on the upper half-cylinder, and here, from the separatrix loop there emerges a stable limit cycle since the saddle index $(P_\varphi' + Q_y')_2 = -2\alpha/s$ is negative [1]. Under a subsequent increase of parameter s along the k-curves the limit cycles monotonically come together. Since there are no limit cycles for the structure in Fig. 1 (*0*), on each k-curve there exists a point with coordinates $s^+(k)$, $\alpha^+(k)$, for which the stable and unstable limit cycles come together, forming a semistable limit cycle. The corresponding structurally unstable bifurcation structure is shown on Fig. 1 (*2—0*). The set of points $s^+(k)$, $\alpha^+(k)$ forms a continuous L^+-curve intersecting each of the k-curves at one point. The sequence of qualitative structures as s increases along the k-curves is shown in Fig. 1 as a sequence of structurally stable structures (*1*), (*2*), (*0*). The structurally unstable structures corresponding to the bifurcation values of the parameters have been denoted by two digits indicating the structurally stable structures which they separate.

Note. The qualitative structures intermediate between structures Fig. 1(*1*) and (*0*) are determined only to within an additional even number of limit cycles girding the cylinder since under a rotation of the field limit cycles can arise from a condensation of the trajectories intersecting the curve $P_\varphi' + Q_y' = 0$, can be separated, and then can once again merge and vanish in other combinations. The logical possibility of such a behavior remains unavoidable. A similar thing cannot occur around the point O_1. Having once arisen the limit cycles cannot vanish since under a further rotation of the field separatrix loops do not arise around the point O_1 and it does not have a change of stability.

4. Location of the bifurcation curves. We remark that the k-curves intersect L and L^+ in a specific sequence and, therefore, L and L^+ do not intersect. Let us show that the curve L^+ lies wholly in the strip $\beta < \alpha < \beta + 1$. We use the comparison system

$$d\varphi / dt = y, \qquad dy / dt = \beta - \sin \varphi - \alpha \qquad (0 < \beta - \alpha < 1) \qquad (4.1)$$

Repeating the arguments of Sect. 2 for the comparison system (2.2), we find that system (1.1) does not have limit cycles for the parameter values $0 < \alpha < \beta$ The quantity $P_\varphi' + Q_y'$ vanishes on the upper half-cylinder only on the straight line $y = s$. If on the cylinder this straight line is a contact-free cycle, then double limit cycles cannot exist [5]. The straight line $y = s$ is a contact-free cycle if

$$\beta - \sin\varphi - 2\alpha s\,\frac{y}{s^2+y^2} = \beta - \sin\varphi - \alpha < 0$$

for all φ, i.e., if $\alpha > \beta + 1$. In the strip $\beta < \alpha < \beta + 1$ the curve L^+ intersects each of the k-curves and, as s decreases, goes from infinity into a point on the α-axis.

Let us trace out the location of curve L. The qualitative pattern of the phase space on any $\varkappa$-curve for small s (for $\alpha = \varkappa s < \beta$) is shown in Fig. 1(*0*). As s increases along the $\varkappa$-curves a monotonic rotation of the field directions takes place, and therefore each $\varkappa$-curve can intersect L not more than once. Consider the comparison system

$$d\varphi \,/\, dt = y, \qquad dy \,/\, dt = \beta - \sin\varphi - 2\varkappa y \tag{4.2}$$

As is well known [6, 7], for each β $(0 < \beta < 1)$ there exists $\varkappa^*$ (β) such that for $\varkappa = \varkappa_1 < \varkappa^*$ (β) the ω-separatrix of the saddle O_2 $(\pi - \arcsin\beta, 0)$ of system (4.2), going out onto the upper half-cylinder, intersects the axis $y = 0$ and goes onto the lower half-cylinder.

Let us write system (1.1) as

$$d\varphi \,/\, dt = y, \qquad dy \,/\, dt = \beta - \sin\varphi - 2\varkappa y s^2\,(s^2 + y^2)^{-1} \tag{4.3}$$

The field of directions of system (4.3) is turned counterclockwise relative to the field of system (4.2), and, therefore, the ω-separatrix of system (4.3) must go onto the lower half-cylinder for arbitrarily large s if $\varkappa < \varkappa^*$ (β). For s sufficiently large and for $\varkappa < \varkappa^*$ (β) unstable and stable limit cycles exist for (4.3) on the upper half-cylinder. For any $\varkappa$ we can choose y_1 such that the expression $\beta - \sin\varphi - 2\varkappa y_1$ preserves sign for all φ. Therefore, for all large s and for (4.3), $dy \,/\, dt < 0$ is fulfilled on the straight line $y = y_1$. But since on the upper half-cylinder the point at infinity is stable (see Sect. 2), an unstable limit cycle exists above the straight line $y = y_1$ for any $\varkappa$. The existence of trajectories winding around the upper half-cylinder from bottom to top, and, consequently, the existence of a stable limit cycle, follows from the location indicated above of the ω-separatrix of the saddle for $\varkappa = \varkappa_1 < \varkappa^*$ (β). We note that $0 < 2\varkappa^* < 1.19$ for $0 < \beta < 1$ and $\varkappa^* \to 0$ as $\beta \to 0$ [8].

The qualitative pattern of the phase space for sufficiently large s on any halfline $\alpha = \varkappa_1 s$ is shown in Fig. 1(*2*). Note that the $\varkappa_1$-curves do not intersect L. Since $\varkappa$-curves intersecting L exist and L goes off to infinity (L intersects each of the k-curves, $0 < k < \infty$), L must have one of the $\varkappa$-curves as an asymptote. It cannot have another $\varkappa$-curve or some straight line parallel to the axis $s = 0$, as a second asymptote since it cannot intersect the k-curve twice. As s decreases the curve L either goes to some point of the axis $s = 0$ or has this axis as its asymptote. Let us show that the first of these possibilities is realized.

For the two systems Fig. 1(*0*) and (*1*), corresponding to the parameter values s_0 and $s_1 < s_0$ curve $y = \sqrt{s_0 s_1}$ (1.2) is the contact curve on the upper half-cylinder. If $\alpha > \beta + 1$, the contact curve is situated above the maximum $y_m = s_1\,(\alpha - \sqrt{\alpha^2 - (\beta+1)^2})\,(\beta+1)^{-1}$ of the lower branch of the isocline of horizontal slopes. Suppose that we have marked, on some straight line $\varphi = \varphi_0$ to the left of saddle O_2, the ordinates η_0 and η_1 of the points of intersection of the straight line with the ω-separatrices for systems (*0*) and (*1*) respectively. In the strip $0 < y < \sqrt{s_0 s_1}$ the vector field of system (*1*) is turned clockwise relative to the vector field of system (*0*) and therefore, $\eta_1 > \eta_0$ for all $s_1 < s_0$. Since the maximum y_m decreases unboundedly as s_1 decreases, $y_m < \eta_1$ for all sufficiently small s_1 and the ω-separatrix of system (*1*) falls into the region above the maximum of the isocline and must wind around the upper half-cylinder. The qualitative structure of the phase space is shown in Fig. 1 (*1*). One unstable limit cycle exists on the upper half-cylinder. Such a structure is realized for any sufficiently small s for any $\alpha \geqslant \beta + 1$ and consequently, the curves L cannot have the axis $s = 0$ as their asymptote.

Note. The bifurcation curves L and L^+ in the $s\beta$-plane were obtained in [2, 3] for various values of parameter α. For small s the curves – the result of calculation and extrapolation – yield a qualitatively incorrect result. The curves cannot go to the origin of the $s\beta$-plane because this is equivalent to the presence of asymptote $s = 0$ for the curves L and L^+ in the αs-plane.

The partitioning of the parameter space foı $\beta = \text{const}$ $(0 < \beta < 1)$ is shown in Fig. 2. The digits *0* — *2* are used to mark the parameter space regions corresponding to the structurally stable structures in Fig. 1 marked by the same digits. The bifurcation curves in Fig. 2, separating the appropriate regions, correspond to the structurally unstable structures in Fig. 1 marked by two digits.

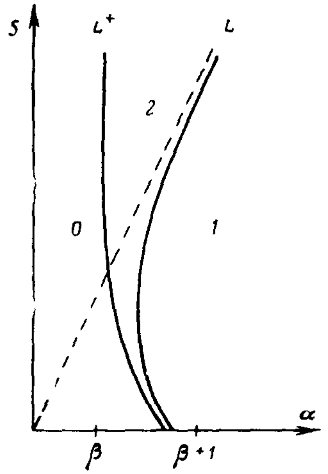

Fig. 2

5. Qualitative patterns and possible bifurcations for $\beta=1$ and $\beta>1$. As β increases up to the value $\beta = 1$ the equilibrium states merge. The structure of the parameter space partitioning for $\beta = 1$ is the same as in Fig. 2. The corresponding structure of the phase space partitioning differs from the structure for the case $0 < \beta < 1$ only in that there is a saddle – node type equilibrium state on the ω-axis. As β increases from the value $\beta = 1$ the saddle – node equilibrium state vanishes for α and s taken from region (*2*) of Fig. 2. A stable limit cycle appears from the α-separatrix of the saddle – node for values of α and s taken from region (*1*). Here the bifurcation curve L vanishes on the parameter plane.

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EVASION CONDITIONS IN A SECOND-ORDER LINEAR DIFFERENTIAL GAME

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Necessary and sufficient evasion conditions in a second-order linear differential game are derived. This paper is closely related with [1 – 4].

1. We consider the second-order system

$$dx/dt = Ax + u - v \tag{1.1}$$

Here x is a two-dimensional phase vector, A is a constant 2×2 matrix, u and v are the controls of the first and second players respectively. We assume that at any instant t

$$u\,(t) \in U, \qquad v\,(t) \in V \tag{1.2}$$

where U is a segment on a plane, not reducing to a point, and V is a bounded closed convex set. The termination of the game means the hitting of system (1.1) onto a certain preassigned point m.

Let us define the notion of evasion. Let the "realization $u\,(\cdot)$" be a measurable time function $u\,(t)$, $t_0 \leqslant t < \infty$, satisfying constraint (1.2) for any t and formed by the first player during the game by some method. We take it that when $t \geqslant t_0$ the second player can collide with any realization $u\,(\cdot)$. The second player is obliged to construct his own control on the feedback principle by means of the discrete scheme $\{v\,[x],\ \Delta\,[x]\}$. The discrete time step $\Delta\,[x] > 0$ defines the size of the semi-interval $t^* \leqslant t < t^* + \Delta\,[x\,[t^*]]$ during which the control v is held constant and depends upon the position $x\,[t^*]$, where it is chosen in accordance with $v\,[x]$.

The discrete scheme $\{v\,[x],\ \Delta\,[x]\}$ is said to be admissible if for any intial position x_0 and for any realization $u(\cdot)$ the switching instants of control v cannot tend from the left to a limit t_* not coinciding with the instant at which system (1.1) hits onto point m. By $T\,[x_0;\ v\,[x], \Delta\,[x],\ u\,(\cdot)]$ we denote the time taken by system (1.1) to go